SOLUTIONS OF SINGULAR CASES OF POINT EXPLOSIONS IN A GAS

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The formulation and the exact solution of self-similar motions associated with strong point-explosions in a gas, the initial density of which is constant or varies as $\rho_1 = Ar^{-\omega}$, was given in Sedov's papers [1,2,3]. Here we will examine the general case of variable initial density. If we introduce dimensionless variables

$$\lambda = \frac{r}{r_2}, \qquad f(\lambda) = \frac{v}{v_2}, \qquad g(\lambda) = \frac{\rho}{\rho_2}, \qquad h(\lambda) = \frac{p}{p_2}$$

(where v represents the speed, ρ the density, p the pressure, $r_2(t)$ the radius of the blastwave, $v_2 = v(r_2, t)$, $\rho_2 = \rho(r_2, t)$, $p_2 = p(r_2, t)$), then the solution of the problem reduces to the integration of the system of equations:

$$\begin{gathered}\frac{df}{d\lambda} = \frac{1}{2}\left[2\gamma(\gamma-1)(\nu-1)\frac{f}{\lambda} + (\nu-\omega)(\gamma+1)\left(f-\frac{\gamma+1}{2}\lambda\right)f\frac{g}{h} - \nu(\gamma^2-1)\right]\times \\ \times\left[2\left(f-\frac{\gamma+1}{2}\lambda\right)^2\frac{g}{h} - \gamma(\gamma-1)\right]^{-1} \\ \frac{dg}{d\lambda} = -g\left[\frac{df}{d\lambda} + (\nu-1)\frac{f}{\lambda} - \frac{\gamma+1}{2}\omega\right]\left(f-\frac{\gamma+1}{2}\lambda\right)^{-1} \\ \frac{dh}{d\lambda} = h\left\{\frac{\gamma+1}{2}\nu - \gamma\left[\frac{df}{d\lambda} + (\nu-1)\frac{f}{\lambda}\right]\right\}\left(f-\frac{\gamma+1}{2}\lambda\right)^{-1}\end{gathered} \qquad (1)$$

Here γ, the ratio of specific heats, exceeds unity and the index ν takes on values 1, 2, or 3 corresponding to plane, cylindrical, or spherical symmetry respectively.

The boundary conditions of the problem state:

$$f(1) = g(1) = h(1) = 1, \qquad f(0) = 0 \qquad (2)$$

Equations (1) have two first integrals (see [3]), the integral of energy and the integral of adiabaticity, which can be represented respectively as

$$\frac{g}{h} = \gamma f^{-2}\left(\frac{f}{\lambda} - \frac{\gamma+1}{2\gamma}\right)\left(\frac{\gamma+1}{2} - \frac{f}{\lambda}\right)^{-1} \tag{3}$$

$$g^{\gamma-1} = \left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - \frac{f}{\lambda}\right)\right]^{1-\frac{\omega\gamma}{\nu}} h^{1-\frac{\omega}{\nu}} \lambda^{\nu-\omega\gamma} \tag{4}$$

when conditions (2) are allowed for. The system of equations (1) can be integrated and its solution, satisfying boundary conditions (2), represented in the form

$$\begin{aligned}
\lambda(F) &= F^{-\delta}\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{-\alpha_2} \times \\
&\times \left[\frac{2(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)-(\gamma+1)\omega}\left(\frac{\gamma+1}{2}\frac{\nu+2-\omega}{\gamma\nu-\nu+2} - F\right)\right]^{-\alpha_1} \\
g(F) &= F^{\omega\delta}\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{\alpha_3+\omega\alpha_2}\left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)\right]^{\alpha_5} \times \\
&\times \left[\frac{2(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)-(\gamma+1)\omega}\left(\frac{\gamma+1}{2}\frac{\nu+2-\omega}{\gamma\nu-\nu+2} - F\right)\right]^{\alpha_4+\omega\alpha_1} \\
h(F) &= F^{\nu\delta}\left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)\right]^{1+\alpha_5} \times \\
&\times \left[\frac{2(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)-(\gamma+1)\omega}\left(\frac{\gamma+1}{2}\frac{\nu+2-\omega}{\gamma\nu-\nu+2} - F\right)\right]^{\alpha_4+(\omega-2)\alpha_1}
\end{aligned} \tag{5}$$

where

$$\delta = \frac{2}{\nu+2-\omega}, \qquad F = \frac{f}{\lambda} \tag{6}$$

$$\alpha_1 = \frac{\gamma+1}{\gamma\nu-\nu+2} - \delta - \alpha_2, \qquad \alpha_2 = \frac{1-\gamma}{2(\gamma-1)+\nu-\omega\gamma} \tag{7}$$

$$\alpha_3 = \frac{\nu-\omega}{2(\gamma-1)+\nu-\omega\gamma} \qquad \alpha_4 = \frac{(\nu-\omega)(\nu+2-\omega)}{2\nu-\nu\gamma-\omega}\alpha_1, \qquad \alpha_5 = \frac{\omega(\gamma+1)-2\nu}{2\nu-\nu\gamma-\omega}$$

In Sedov's solutions [3], the related parameter V rather than F was used:

$$V = \frac{4}{(\nu+2-\omega)(\gamma+1)} F$$

When the solution may be extended to the center of symmetry [3], then the parametric variable F is constrained according to

$$1 \geqslant F \geqslant \frac{\gamma+1}{2\gamma}$$

Hereafter we shall examine only this case. The solutions (5) possess singularities with respect to the parameter γ when either the coefficients on the right side of (5) or the quantities $\alpha_i (i = 1, 2, \ldots 5)$, defined by (7), approach infinity. These cases we shall call γ-singular. In all these cases one cannot use the solutions in the form (5) - (7) for computations of functions $f(\lambda)$, $g(\lambda)$, and $h(\lambda)$. Therefore it is necessary to seek new representations of the solutions for these singular cases.

The case of the infinite coefficients in (5), corresponding to the value of ω

$$\omega_1 = \frac{3\nu - 2 + \gamma(2-\nu)}{\gamma+1}$$

has been solved earlier [3] and is given by

$$f = \lambda, \qquad g = \lambda^{\nu-2}, \qquad h = \lambda^{\nu} \tag{8}$$

Hereafter the form of the solution is found in two other cases, namely when

$$\omega \to \omega_2 = \frac{2(\gamma-1)+\nu}{\gamma} \qquad (\alpha_1, \alpha_2, \alpha_3, \alpha_4 \to \infty)$$

and when

$$\omega \to \omega_3 = \nu(2-\gamma) \qquad (\alpha_4, \alpha_5 \to \infty)$$

The form of the solution may be obtained either by performing the limiting process $\omega \to \omega_j (j = 2, 3)$ in equations (5) - (9), or directly from the differential equations (1), with ω replaced by ω_j. The second approach is simpler.

With the aid of the energy integral (3) the equations (1) can be transformed as follows:

$$\frac{df}{d\lambda} = -\frac{f}{\lambda}\left[\frac{\gamma(\gamma-1)(\nu-1)}{\gamma+1}\left(\frac{f}{\lambda}\right)^2 + \frac{\omega\gamma+\nu-2\nu\gamma}{2}\frac{f}{\lambda} + \frac{(\nu-\omega)(\gamma+1)}{4}\right]\times$$
$$\times\left[\gamma\left(\frac{f}{\lambda}\right)^2 - (\gamma+1)\frac{f}{\lambda} + \frac{\gamma+1}{2}\right]^{-1} \tag{9}$$

$$\frac{dg}{d\lambda} = -g\left[\frac{df}{d\lambda} + (\nu-1)\frac{f}{\lambda} - \frac{\gamma+1}{2}\omega\right]\left(f - \frac{\gamma+1}{2}\lambda\right)^{-1} \tag{10}$$

Substituting $f = \lambda F$ (6) and $\omega = \omega_2$ in equation (9) and integrating, with due regard for boundary conditions (2), we obtain

$$\lambda(F) = F^{-2\gamma\delta_2}\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{(\gamma-1)\delta_2}\exp\left[-(\gamma+1)\delta_2\frac{1-F}{F-(\gamma+1)/2\gamma}\right]$$
$$\delta_2 = \frac{1}{\gamma\nu - \nu + 2} \tag{11}$$

Further, by means of the integrals (3) and (4) we find

$$g(F) = F^{2(2\gamma-2+\nu)\delta_2}\left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)\right]^{[\nu-2(\gamma+1)]\delta_2}\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{(4-\nu-2\gamma)\delta_2}\times$$
$$\times\exp\left[2(\gamma+1)\delta_2\frac{1-F}{F-(\gamma+1)/2\gamma}\right] \tag{12}$$

$$h(F) = F^{2\nu\gamma\delta_2}\left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)\right]^{\gamma(\nu-2)\delta_2}\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{-\gamma\nu\delta_2} \tag{13}$$

Formulas (6), (11), (12), and (13) furnish the solution in the second singular case $\omega = \omega_2$.

Let us now examine the third singular case $\omega = \omega_3 = \nu(2 - \gamma)$. Then it is easy to establish the form of the function $g(F)$ from equations (10), (6), and (9). Further, utilizing (3) and (4), one determines $\lambda(F)$ and $h(F)$. The solutions have the form

$$\lambda(F) = F^{-2\delta_2}\left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)\right]^{-\gamma\delta_2}\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{\delta_2}$$

$$g(F) = F^{2\nu(2-\gamma)\delta_2}\left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)\right]^{(\nu\gamma-\nu-2)\delta_2}\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{\nu(\gamma-1)\delta_2} \times$$

$$\times \exp\left[-\nu\gamma(\gamma+1)\delta_2\frac{1-F}{1/2(\gamma+1) - F}\right] \tag{14}$$

$$h(F) = F^{2\nu\delta_2}\left[\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)\right]^{2(\nu\gamma-\nu-\gamma)\delta_2} \exp\left[-\nu\gamma(\gamma+1)\delta_2\frac{1-F}{1/2(\gamma+1) - F}\right]$$

As a consequence of self similarity [2,3] the position of the blast wave and its speed depend on time as follows:

$$r_2(t) = \left(\frac{E_0}{\alpha A}\right)^{1/2\delta} t^{\delta}, \qquad c(t) = \delta\,\frac{r_2(t)}{t} \tag{15}$$

Here E_0 represents the energy of the explosion and α is a function of γ, ν, and ω determined by the conservation of energy

$$E_0 = \sigma_\nu \int_0^{r_2} \left(\frac{\rho v^2}{2} + \frac{p}{\gamma-1}\right) r^{\nu-1}\,dr \tag{16}$$

In terms of the dimensionless variables, (16) is written:

$$\alpha(\gamma, \nu, \omega) = \frac{2\sigma_\nu \delta^2}{(\gamma^2-1)} \int_0^1 (h + gf^2)\ \lambda^{\nu-1}\,d\lambda \tag{17}$$

$$\sigma_\nu = 2\pi(\nu-1) + (\nu-2)(\nu-3)$$

It is clear that for the first singular case with solution (8) the function $\alpha(\gamma, \nu, \omega)$ simplifies to

$$\alpha(\gamma, \nu, \omega_1) = \frac{2\sigma_\nu}{\nu}\,\frac{\gamma+1}{\gamma-1}\left(\frac{1}{\gamma\nu - \nu + 2}\right)^2$$

The magnitudes of the parameters at the blast wave are found from the expressions for strong shock waves:

$$v_2 = \frac{2}{\gamma+1}c, \qquad \rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1, \qquad p_2 = \frac{2}{\gamma+1}\rho_1 c^2 \tag{18}$$

The expressions (11), (12), (13), (6), (14), (15), (17), and (18) give the full solution for the singular cases $\omega = \omega_2$ and $\omega = \omega_3$.

As previously mentioned, each singular solution can be obtained by an appropriate limiting process. For the sake of simplicity we shall show how this is done in the third singular case with $\omega_3 = 0$. Consequently $\gamma \to 2$ for all three values of the index ν. Consider the limit of the function

$$g(F) = \left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{\alpha_3} \left[\frac{\frac{2(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)}\left(\frac{(\gamma+1)(\nu+2)}{2(\nu\gamma-\nu+2)} - F\right)}{\frac{2}{\gamma-1}\left(\frac{\gamma+1}{2} - F\right)}\right]^{-\alpha_5}$$
$$\times \left[\frac{2(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)}\left(\frac{(\gamma+1)(\nu+2)}{2(\nu\gamma-\nu+2)} - F\right)\right]^{\alpha_4+\alpha_5}$$

obtained from equation (5). For $\gamma \to 2$, we have:

$$\lim_{\gamma\to 2} g(F) = \lim_{\gamma\to 2}\left\{\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{\alpha_3}\right\} \times$$
$$\times \lim_{\gamma\to 2}\left\{\left[\frac{\frac{(\gamma-1)(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)}\left(\frac{(\gamma+1)(\nu+2)}{2(\nu\gamma-\nu+2)} - F\right)}{\frac{\gamma+1}{2} - F}\right]^{-\alpha_6}\right\} \times$$
$$\times \lim_{\gamma\to 2}\left\{\left[\frac{2(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)}\left(\frac{(\gamma+1)(\nu+2)}{2(\nu\gamma-\nu+2)} - F\right)\right]^{\alpha_4+\alpha_6}\right\}$$

After a simplification of the second bracket in this expression we arrive at

$$\lim_{\gamma\to 2} g(F) = \lim_{\gamma\to 2}\left\{\left[\frac{2\gamma}{\gamma-1}\left(F - \frac{\gamma+1}{2\gamma}\right)\right]^{\alpha_3}\right\} \lim_{\substack{x\to\infty \\ \gamma\to 2}}\left(1 + \frac{1}{x}\right)^{-\frac{2\nu(\gamma+1)}{\nu+2}\frac{1-F}{\frac{\gamma+1}{2}-F}x} \times$$
$$\times \lim_{\gamma\to 2}\left\{\left[\frac{2(\nu\gamma-\nu+2)}{3\nu-2-\gamma(\nu-2)}\left(\frac{(\gamma+1)(\nu+2)}{2(\nu\gamma-\nu+2)} - F\right)\right]^{\alpha_4+\alpha_6}\right\},$$

where the notation is used

$$x = (\nu+2)(2-\gamma)^{-1}\left(\frac{\gamma+1}{2} - F\right)[\nu(\gamma+1)(1-F)]^{-1}$$

Going to the limit we find:

$$g(F) = \left[4\left(F - \frac{3}{4}\right)\right]^{\frac{\nu}{\nu+2}} \left[2\left(\frac{3}{2} - F\right)\right]^{\frac{\nu-2}{\nu+2}} \exp\left(-\frac{6\nu}{\nu+2}\,\frac{1-F}{3/2 - F}\right)$$

One can also determine $h(F)$ and $\lambda(F)$ by the limiting process, the latter determination being altogether trivial. However, this is unnecessary because with the aid of the integrals (3) and (4) and the preceding expression for $g(F)$ one can obtain the full solution, which becomes equivalent to (14) for $\gamma = 2$. The limiting process in the case of arbitrary non-zero values of $\omega_3(\gamma, \nu)$ goes through analogously. Similarly one can establish the solutions for $\omega \to \omega_2$.

The present investigation shows that the solution of the problem of strong explosions are continuous in γ, as one would expect from the form of the basic differential equations (1). In particular, the investigation removes all doubts concerning the solutions for $\gamma \to 2$.

In this connection, we note that for this problem reference [4] (pp. 556-557) contains errors in the description of the pressure variation near the center of symmetry ($\lambda = 0$, $F = (\gamma + 1)/2\gamma$) when $\omega = 0$ and $\gamma \to 2$. In this case we find the value of h at the center from equation (14):

$$h_0 = 3^{\frac{4(\nu-1)}{\nu+2}} 4^{\frac{2-3\nu}{\nu+2}} e^{-\frac{2\nu}{\nu+2}} \neq 0$$

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